PENETRATION OF RIGID DIES INTO A HALF-SPACE WITH POWER-LAW STRAIN-HARDENING AND WITH NONLINEAR CREEP OF THE MATERIAL

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PMM Vol.26, No.3, 1962, pp. 481-491

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(Received February 20, 1962)

In [1,2] Arutiunian examines the plane contact problem for the case of a material which strain-hardens according to a power-law and which possesses the property of nonlinear creep of the hereditary type. Using the ideas put forward in [1,2], the present paper considers the three-dimensional contact problem on the basis of the same assumptions concerning the properties of the material.

1. Concentrated force applied to the boundary of a half-space with power-law strain-hardening of the material. Suppose that a concentrated force P is applied normally to the boundary of a half-space $z \geqslant 0$.

Using a system of spherical coordinates (see Figure), we see that in view of the axial symmetry of the problem

$$u_\varphi = 0, \qquad \varepsilon_{R\varphi} = \varepsilon_{\theta\varphi} = 0, \qquad \tau_{R\varphi} = \tau_{\theta\varphi} = 0$$

and the nonzero displacements u_R, u_θ, the strains ε_R, ε_θ, $\varepsilon_{R\theta}$, ε_φ and the stresses σ_R, σ_θ, $\tau_{R\theta}$, σ_φ are independent of φ.

We shall assume that the material is incompressible

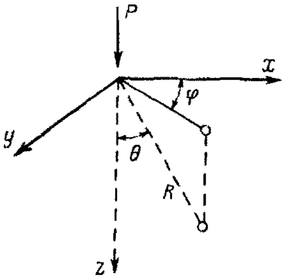
$$\varepsilon_R + \varepsilon_\theta + \varepsilon_\varphi = 0 \tag{1.1}$$

and that the strain-hardening is given by (1.2)

$$\sigma_{ij}' = A\Gamma^{\mu-1}\varepsilon_{ij} \qquad \left(\Gamma^2 = \frac{2}{3}\left[(\varepsilon_R - \varepsilon_\theta)^2 + (\varepsilon_\theta - \varepsilon_\varphi)^2 + (\varepsilon_\varphi - \varepsilon_R)^2\right] + 4\varepsilon_{R\theta}^2\right)$$

Here Γ is the intensity of shear strain; σ_{ij}' is the stress deviator, A and μ are constants of the material, where $0 < \mu \leqslant 1$.

The stresses must satisfy the equations of equilibrium



$$\frac{\partial \sigma_R}{\partial R} + \frac{1}{R}\frac{\partial \tau_{R\theta}}{\partial \theta} + \frac{2\sigma_R - \sigma_\theta - \sigma_\varphi}{R} + \frac{\cot\theta}{R}\tau_{R\theta} = 0 \qquad (1.3)$$

$$\frac{1}{R}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{R\theta}}{\partial R} + \frac{\cot\theta}{R}(\sigma_\theta - \sigma_\varphi) + \frac{3}{R}\tau_{R\theta} = 0 \qquad (1.4)$$

They must also satisfy the boundary conditions

$$\tau_{R\theta} = 0, \quad \sigma_\theta = 0 \quad \text{at} \quad \theta = \frac{\pi}{2} \text{ and } R \neq 0 \qquad (1.5)$$

and must be statically equivalent to the force P

$$\int_0^{2\pi} d\varphi \int_0^{\pi/2} (\sigma_R \cos\theta - \tau_{R\theta}\sin\theta) R^2 \sin\theta \, d\theta + P = 0 \qquad (1.6)$$

The strains must satisfy the compatibility conditions [3]

$$\begin{gathered}
\frac{1}{R}\frac{\partial^2 \varepsilon_\varphi}{\partial \theta^2} + \frac{2\cot\theta}{R}\frac{\partial \varepsilon_\varphi}{\partial \theta} + \frac{\partial}{\partial R}(\varepsilon_\theta + \varepsilon_\varphi) - \frac{\cot\theta}{R}\frac{\partial \varepsilon_\theta}{\partial \theta} + \frac{2}{R}(\varepsilon_\theta - \varepsilon_R) - \\
- \frac{2}{R}\left(\frac{\partial \varepsilon_{R\theta}}{\partial \theta} + \varepsilon_{R\theta}\cot\theta\right) = 0 \\
\frac{\partial^2(\varepsilon_\varphi R)}{\partial R^2} - \frac{\partial \varepsilon_R}{\partial R} + \frac{\cot\theta}{R}\frac{\partial \varepsilon_R}{\partial \theta} - \frac{2\cot\theta}{R}\frac{\partial(\varepsilon_{R\theta}R)}{\partial R} = 0 \\
\frac{\partial^2(\varepsilon_\theta R)}{\partial R^2} + \frac{1}{R}\frac{\partial^2 \varepsilon_R}{\partial \theta^2} - \frac{\partial \varepsilon_R}{\partial R} - \frac{2}{R}\frac{\partial^2(\varepsilon_{R\theta}R)}{\partial R \partial \theta} = 0 \qquad (1.7) \\
\frac{1}{R}\frac{\partial \varepsilon_R}{\partial \theta} - \frac{\partial^2 \varepsilon_\varphi}{\partial R \partial \theta} - \cot\theta \frac{\partial \varepsilon_\varphi}{\partial R} + \cot\theta \frac{\partial \varepsilon_\theta}{\partial R} - \frac{2}{R}\varepsilon_{R\theta} = 0
\end{gathered}$$

The relation between the displacements is given by the formulas

$$\begin{gathered}
\varepsilon_R = \frac{\partial u_R}{\partial R}, \qquad \varepsilon_\theta = \frac{1}{R}\frac{\partial u_\theta}{\partial \theta} + \frac{u_R}{R}, \qquad \varepsilon_\varphi = \frac{u_\theta \cot\theta}{R} + \frac{u_R}{R} \qquad (1.8) \\
2\varepsilon_{R\theta} = \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R} + \frac{1}{R}\frac{\partial u_R}{\partial \theta}
\end{gathered}$$

Note the particular cases of the solution of this problem. If $\mu = 1$ (an elastic incompressible material) [3]

$$\sigma_R = -\frac{3}{2}\frac{P}{\pi}\frac{\cos\theta}{R^2}, \qquad \sigma_\theta = \tau_{R\theta} = \sigma_\varphi = 0 \qquad (1.9)$$

$$\varepsilon_R = -\frac{P}{2\pi G}\frac{\cos\theta}{R^2}, \qquad \varepsilon_\theta = \varepsilon_\varphi = -\frac{1}{2}\varepsilon_R, \qquad \varepsilon_{R\theta} = 0$$
$$u_R = \frac{P}{2\pi G}\frac{\cos\theta}{R}, \qquad u_\theta = -\frac{P}{4\pi G}\frac{\sin\theta}{R}$$

where G is the shear modulus. If $\mu = 2/3$ the solution assumes the form [4]

$$\sigma_R = -\frac{P}{\pi}\frac{1}{R^2}, \qquad \sigma_\theta = \tau_{R\theta} = \sigma_\varphi = 0$$
$$\varepsilon_R = -3^{0.75}\, 2\sqrt{2}\left(\frac{P}{\pi A}\right)^{1.5}\frac{1}{R^3}, \qquad \varepsilon_\theta = \varepsilon_\varphi = -\frac{1}{2}\varepsilon_R, \qquad \varepsilon_{R\theta} = 0 \tag{1.10}$$
$$u_R = 3^{0.75}\sqrt{2}\left(\frac{P}{\pi A}\right)^{1.5}\frac{1}{R^2}, \qquad u_\theta = 0$$

It can easily be shown that a solution in the form

$$\sigma_R = \frac{F(\theta)}{R^2}, \qquad \sigma_\theta = \tau_{R\theta} = \sigma_\varphi = 0$$

exists only for these values of μ.

It follows from conditions (1.6) that the stresses σ_R and $\tau_{R\theta}$ have a singularity $1/R^2$ at $R = 0$. However, condition (1.6) tells us nothing of the behavior of the stresses σ_θ and σ_φ at $R = 0$.

We make the following assumption. We assume that the variables R and θ in the expressions for the stresses are separable. This assumption is prompted by the homogeneity of each of the groups of Equations (1.1), (1.2), (1.3), (1.4) and (1.7) which must be satisfied. Finally, it is clear that the assumption made is valid for $\mu = 1$ and $\mu = 2/3$. Then from condition (1.6) and from the differential equations of equilibrium, it follows that the stresses have the form

$$\sigma_{ij} = F_{ij}(\theta) R^{-2} \tag{1.11}$$

Then, on the basis of Equations (1.2)

$$\varepsilon_{ij} = e_{ij}(\theta) R^{-2m}, \qquad m = \frac{1}{\mu} \tag{1.12}$$

We take the displacements in the form

$$u_R = \xi(\theta) R^{-2m+1}, \qquad u_\theta = \eta(\theta) R^{-2m+1} \tag{1.13}$$

The condition of incompressibility (1.1) enables us to express $\xi(\theta)$ in terms of $\eta(\theta)$

$$\xi(\theta) = \cot\theta\,\zeta(\theta) + \zeta'(\theta), \qquad \zeta(\theta) = \frac{\eta(\theta)}{2m-3} \tag{1.14}$$

We shall now consider the case when $\mu \neq 2/3$. (If $\mu = 2/3$ the solution is given by Formulas (1.10.) The displacements (1.13) to (1.14) correspond to "strains" $\varepsilon_{ij}(\theta)$ given by

$$e_R = (-2m+1)(\cot\theta\zeta + \zeta'), \quad e_\theta = \cot\theta\zeta + 2(m-1)\zeta'$$
$$e_\varphi = 2(m-1)\cot\theta\zeta + \zeta' \tag{1.15}$$
$$2e_{R\theta} = -[2m(2m-3) + 1 + \cot^2\theta]\zeta + \cot\theta\zeta' + \zeta$$

We will show that the expressions for the strains (1.12) for values of $\varepsilon_{ij}(\theta)$ given by (1.15) (and consequently the expressions for the displacements (1.13) to (1.14) as well) are general expressions, provided the variables are separable and the material is incompressible. To do so we substitute (1.12) into conditions (1.7). After eliminating ε_R by means of the condition of incompressibility, we obtain

$$X_1 \equiv e_\varphi'' + 2\cot\theta e_\varphi' - \cot\theta e_\theta' - 2e_{R\theta}' - 2(m-1)e_\varphi - 2(m-2)e_\theta - 2\cot\theta e_{R\theta} = 0$$
$$X_2 \equiv \cot\theta(e_\varphi' + e_\theta') - 4m(m-1)e_\varphi + 2me_\theta - 2(2m-1)\cot\theta e_{R\theta} = 0$$
$$X_3 \equiv e_\varphi'' + e_\theta'' - 2(2m-1)e_{R\theta}' + 2me_\varphi - 4m(m-1)e_\theta = 0$$
$$X_4 \equiv (2m-1)e_\varphi' - e_\theta' + 2m\cot\theta(e_\varphi - e_\theta) - 2e_{R\theta} = 0 \tag{1.16}$$

The system of Equations (1.16) is equivalent to

$$Y_1 \equiv -X_1 + X_4\cot\theta + X_4' = 0, \qquad Y_3 \equiv -X_2 + (2m-1)X_4\cot\theta = 0$$
$$Y_2 \equiv -X_3 + (2m-1)X_4' = 0, \qquad X_4 = 0$$

It can be verified that the following identities hold:

$$2mY_1 \equiv Y_2 + Y_3, \qquad 2m(Y_2 - Y_3) \equiv (Y_3\,\mathrm{tg}\,\theta)'$$

The system of equations $Y_i = 0$ ($i = 1, 2, 3$) is therefore equivalent to the equation $Y_3 = 0$. Consequently, the system (1.16) is equivalent to the equations $Y_3 = 0$ and $X_4 = 0$. In expanded form the equation $Y_3 = 0$ has the form

$$2(m-1)\cot\theta e_\varphi' + [2(m-1) + (2m-1)\cot^2\theta]e_\varphi - \cot\theta e_\theta' - [1 + (2m-1)\cot^2\theta]e_\theta = 0 \tag{1.17}$$

Integrating (1.17) and making use of the equation $X_4 = 0$, we find that

$$e_\theta = C\cos\theta\sin^{-2m+1}\theta + 2(m-1)e_\varphi - (2m-1)(2m-3)\cos\theta\sin^{-2m+1}\theta \times \\ \times \int_0^\theta e_\varphi \sin^{2m-2}\theta d\theta \tag{1.18}$$

$$2e_{R\theta} = -C\cos 2\theta\sin^{-2m}\theta + e_\varphi' - (2m-3)\cot\theta e_\varphi + (2m-1)(2m-3) \times \\ \times \cos 2\theta \sin^{-2m}\theta \int_0^\theta e_\varphi \sin^{2m-2}\theta d\theta$$

where C is a constant of integration.

Now consider the third equation of (1.15) as an equation in ζ. Solving this equation and eliminating ζ in the second and fourth equations of (1.15) by expressing it in terms of e_φ, we obtain expressions for e_θ and $e_{R\theta}$ identical with (1.18). This substantiates the statement made above concerning the generality of Formulas (1.15) and (1.13) to (1.14).

We shall define $s_{ij}(\theta)$ and $s(\theta)$ as quantities which depend only on θ in the expressions for the stress deviator σ_{ij}' and the mean pressure σ

$$\sigma_{ij}' = s_{ij}(\theta) R^{-2}, \qquad \sigma = s(\theta) R^{-2} \tag{1.19}$$

Equations (1.2) can then be written in the form

$$s_{ij} = A g^{\mu-1} e_{ij} \tag{1.20}$$

where

$$g^2 = \tfrac{2}{3}[(e_R - e_\theta)^2 + (e_\theta - e_\varphi)^2 + (e_\varphi - e_R)^2] + 4e_{R\theta}^2, \qquad \Gamma = g(\theta) R^{-2m} \tag{1.21}$$

With the aid of (1.19) we can write the equations of equilibrium (1.3), (1.4) in the form

$$\begin{aligned} s_{R\theta}' + \cot\theta s_{R\theta} - (s_\theta + s_\varphi) - 2s = 0 \\ s' + s_\theta' + s_{R\theta} + \cot\theta(s_\theta - s_\varphi) = 0 \end{aligned} \tag{1.22}$$

The first of the equations expresses the "mean pressure" s in terms of the "components of the stress deviator" s_{ij}. Eliminating s from the second equation, we obtain an equation which contains only s_{ij}.

$$s_{R\theta}'' + \cot\theta s_{R\theta}' + (1-\cot^2\theta)s_{R\theta} + s_\theta' - s_\varphi' + 2\cot\theta(s_\theta - s_\varphi) = 0 \tag{1.23}$$

Substituting Formulas (1.20) into Equation (1.23) yields

$$g^{\mu-1}[e_{R\theta}'' + \cot\theta e_{R\theta}' + (1-\cot^2\theta)e_{R\theta} + e_\theta' - e_\varphi' + 2\cot\theta(e_\theta - e_\varphi)] + \\ + (\mu-1)g^{\mu-3}\{2gg'e_{R\theta}' + [(\mu-2)g'^2 + gg'' + \cot\theta gg']e_{R\theta} + \\ + gg'(e_\theta - e_\varphi)\} = 0 \tag{1.24}$$

If we now substitute into (1.24) values of e_{ij} given by (1.15) we obtain a nonlinear ordinary differential equation of the fourth order in the function $\zeta(\theta)$

$$U\zeta = 0 \tag{1.25}$$

Here U denotes the appropriate differential operator. The boundary conditions (1.5) mean that

$$s_{R\theta} = 0, \qquad s_\theta + s = 0 \quad \text{at} \quad \theta = \tfrac{1}{2}\pi \tag{1.26}$$

Eliminating s with the aid of the first equation of (1.22), we obtain

$$s_{R\theta} = 0, \qquad s_{R\theta}' + s_\theta - s_\varphi = 0 \quad \text{at} \quad \theta = \tfrac{1}{2}\pi \tag{1.27}$$

or, after substituting Formulas (1.20)

$$e_{R\theta} = 0, \qquad e_{R\theta}' + e_\theta - e_\varphi = 0 \quad \text{at} \quad \theta = \tfrac{1}{2}\pi \tag{1.28}$$

If we now substitute values of e_{ij} given by (1.15) into (1.28) we obtain the following boundary conditions for the function $\zeta(\theta)$:

$$\zeta'' + [1 + 2m(2m - 3)]\,\zeta = 0, \qquad \zeta''' + 2(2m^2 - 5m + 4)\,\zeta' = 0 \quad \text{at} \quad \theta = \tfrac{1}{2}\pi \tag{1.29}$$

Also, from the condition that $u_\theta = 0$ when $\theta = 0$ we have that

$$\zeta = 0 \quad \text{at} \quad \theta = 0 \tag{1.30}$$

With the aid of the first equation of (1.22) condition (1.6) can be reduced to the form

$$\int_0^{\pi/2} [s_{R\theta} \sin\theta + 3(s_\theta + s_\varphi)\cos\theta] \sin\theta\, d\theta = \frac{P}{\pi} \tag{1.31}$$

If we substitute (1.20) and (1.15) into this expression we obtain the condition

$$V[\zeta] = \int_0^{\pi/2} g^{\mu-1} [e_{R\theta} \sin\theta + 3(e_\theta + e_\varphi)\cos\theta] \sin\theta\, d\theta = \frac{P}{\pi A} \tag{1.32}$$

Thus the problem has been reduced to one of finding the function $\zeta(\theta)$ which satisfies the ordinary fourth-order differential equation (1.25) and the conditions (1.29), (1.30) and (1.32).

For $\mu = 1$ Equation (1.25) is linear and easily integrable. In this case conditions (1.29), (1.30) and (1.32) select from the general solution the function $\zeta(\theta)$ corresponding to (1.9). If $\mu \neq 1$, Equation (1.25) becomes an extremely complex nonlinear differential equation which it was found impossible to integrate. However, for the present purposes it will be sufficient to carry out an analysis of the dependence of the solution to the problem (1.25), (1.29), (1.30), (1.32) on parameters.

We note that the left-hand side of (1.25) represents a homogeneous function of order μ in ζ, ζ', ..., ζ^{IV}. The boundary conditions are also homogeneous, whilst in equalities (1.25), (1.29) and (1.30), the only parameter of the problem to appear is μ. Therefore, if $\zeta_0 = \zeta_0(\theta, \mu)$ is some particular solution of the problem (1.25), (1.29), (1.30), then $\zeta = B\zeta_0$, where B is an arbitrary constant, is also a solution of this problem. The constant B can be found from condition (1.32)

$$B = D^{-m}(\mu)\left(\frac{P}{\pi A}\right)^m, \qquad D(\mu) = V[\zeta_0]$$

Consequently, the displacements are given by

$$\begin{aligned} u_R &= D^{-m}(\mu)[\cot\theta\zeta_0(\theta, \mu) + \zeta_0'(\theta, \mu)]\left(\frac{P}{\pi A}\right)^m R^{-2m+1} \\ u_\theta &= (2m-3)\, D^{-m}(\mu)\,\zeta_0(\theta, \mu)\left(\frac{P}{\pi A}\right)^m R^{-2m+1} \end{aligned} \tag{1.33}$$

The settlement of points on the boundary of the half-space is expressed by

$$w(x, y) = -u_\theta\big|_{\theta=\frac{\pi}{2}} = c(\mu)\left(\frac{P}{A}\right)^m r^{-2m+1}, \qquad r = \sqrt{x^2+y^2} \tag{1.34}$$

$$c(\mu) = -(2m-3)\,[\pi D(\mu)]^{-m}\,\zeta_0(\pi/2, \mu), \; c(1) = 1/4\pi$$

Here $c(\mu)$ is a constant dependent only on μ. Expression (1.34) can be rewritten as

$$Ac^{-\mu}w^{\mu} = \frac{P}{r^{2-\mu}} \tag{1.35}$$

2. Penetration of a die into a half-space with power-law strain-hardening. Suppose that a rigid die is pressed without friction into the half-space $z \geqslant 0$. We shall assume that the properties of the material are defined by Equations (1.1) and (1.2). The problem is to find the settlement of the die and the pressure distribution over the area of contact S. The settlement of points on the area of contact is

$$w(x, y) = \alpha x + \beta y + w_0 - \varphi(x, y) \tag{2.1}$$

where $z = -\varphi(x, y)$ is the equation of the surface of the die at the moment of contact with the half-space, $\alpha x + \beta y + w_0$ is the unknown rigid-body displacement.

In order to derive an approximate solution to this problem we use the method suggested by Arutiunian [1] and apply the principle of addition of "generalized displacements" w^{μ}. Then, denoting the pressure on the area of contact by $p(x, y)$ and making use of Formula (1.35), we obtain for $p(x, y)$ an integral equation* analogous to the equation in the corresponding plane prbblem [1]

$$\iint_{(S)} \frac{p(x_1, y_1)\, dx_1 dy_1}{\left(\sqrt{(x-x_1)^2+(y-y_1)^2}\right)^{2-\mu}} = Ac^{-\mu} w^{\mu}(x, y) \tag{2.2}$$

in which the function $w(x, y)$ is given by (2.1). In order to find the constants α, β and w_0 we have the equations of equilibrium of the die

$$\iint_{(S)} p\, dx dy = P \tag{2.3}$$

$$\iint_{(S)} py\, dx dy = M_x, \qquad \iint_{(S)} px\, dx dy = -M_y \tag{2.4}$$

where P, M_x, M_y are the given force and moment components applied to the die. If the die has a smooth shape, then in order to determine the value of S we apply the condition that the pressure vanishes at the boundary of the area S.

If an axially symmetrical die is pressed into a half-space by a force P, then the function $w(x, y)$ in (2.2) is replaced by the function

$$w(r) = w_0 - \varphi(r) \tag{2.5}$$

and only (2.3) is retained from conditions (2.3) and (2.4).

Equation (2.2) is a linear Fredholm integral equation of the first kind with kernel having a point of non-essential singularity ($1 \leqslant 2 - \mu < 2$).

It is interesting to note that an equation analogous to (2.2) is

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* If the area of contact of two bodies with different constants A_1, A_2 and the same power μ is sufficiently small the absolute term in Equation (2.2) is replaced by the function

$$[c\,(A_1^{-m} + A_2^{-m})]^{-\mu}\,[\alpha x + \beta y + w_0 - \varphi_1(x, y) - \varphi_2(x, y)]^{\mu}$$

where $z = \varphi_1(x, y)$, $z = -\varphi_2(x, y)$ are the equations of the surfaces of the bodies at the moment of contact.

obtained in the problem of the penetration of a rigid die into a nonhomogeneous elastic half-space with a Young's modulus

$$E = E_n z^n, \qquad n, E_n = \text{const} \qquad (0 < n \leqslant 1)$$

and with a Poisson's ratio $\nu = 1/(2 + n)$; in this case [5]

$$\iint\limits_{(S)} \frac{p(x_1, y_1)\, dx_1 dy_1}{\left(\sqrt{(x - x_1)^2 + (y - y_1)^2}\right)^{1+n}} = \gamma w(x, y) \qquad (\gamma = \text{const}) \qquad (2.6)$$

Making use of this analogy, we can formulate the following result. Suppose that a die with a flat base is pressed into a half-space with a force P. Then the absolute terms in Equations (2.2) and (2.6) are the constants $Ac^{-\mu}w_0^{\mu}$ and γw_0, respectively. We define $p_1(x, y)$ as the solution of Equation (2.2) with an absolute term equal to unity. Then the pressure under the die in the case of a half-space which strain-hardens is given by

$$p(x, y) = Ac^{-\mu}w_0^{\mu}p_1(x, y)$$

and in the case of a nonhomogeneous elastic half-space (when $n = 1 - \mu$), by

$$p(x, y) = \gamma w_0 p_1(x, y)$$

If we now eliminate the constants $Ac^{-\mu}w_0^{\mu}$ and γw_0 in terms of P with the aid of (2.3), then in the case under discussion the laws governing the pressure distribution under a die pressing on a strain-hardening half-space and under the same die pressing on a nonhomogeneous elastic half-space ($n = 1 - \mu$) are identical. Note, however, that the equations relating the settlement w_0 with the force P are different.

In [5] Rostovtsev has derived a solution to Equation (2.6) for a die elliptical in plan with a polynomial absolute term (a generalization of Shtaerman's theorem for a homogeneous elastic material). He has also derived a solution for a circular area of contact.

Let us consider the penetration of a die with a plane elliptical base under the action of a force P. Making use of the results of [5] we find that

$$p(x, y) = \frac{(2 - \mu)P}{2\pi ab}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-\mu/2}, \qquad P = Ac^{-\mu}\frac{\sin \pi\mu / 2a^{1-\mu}b}{(2 - \mu)K} w_0^{\mu} \quad (2.7)$$

where b, a are the semi-axes of the ellipse ($a \leqslant b$)

$$K = \int_0^{\pi/2} \frac{d\alpha}{(\sqrt{1 - e^2 \sin^2 \alpha})^{\mu}} \qquad \left(e^2 = 1 - \frac{a^2}{b^2}\right) \tag{2.8}$$

In particular, for a die with a plane circular base (a being its radius)

$$p(r) = \frac{(2-\mu) P}{2\pi a^{2-\mu}} \frac{1}{(\sqrt{a^2 - r^2})^{\mu}}, \qquad P = Ac^{-\mu} \frac{2 \sin \pi\mu / 2}{\pi (2-\mu)} w_0^{\mu} \tag{2.9}$$

If $\mu = 1$, Formulas (2.7) become the corresponding well-known formulas for the case of a homogeneous elastic material. Note that the pressure distribution (2.9) is analogous to the pressure distribution in the problem of the penetration of a rectangular die under conditions of plane deformation [1].

In the case of an axially-symmetrical die under the action of a force P, we have, on the basis of the results in [5], that

$$p(r) = Ac - \mu \frac{1}{\pi^2} \sin \frac{\pi\mu}{2} \left[\frac{\psi(a)}{(a^2 - r^2)^{\mu/2}} - \int_r^a \frac{\psi'(u)\, du}{(u^2 - r^2)^{\mu/2}} \right] \tag{2.10}$$

$$\psi(u) = w_0^{\mu} + u^{\mu} \int_0^u \frac{[w^{\mu}(v)]'\, dv}{(u^2 - v^2)^{\mu/2}} \tag{2.11}$$

If the die has a smooth shape, then $p(a) = 0$ and consequently

$$\psi(a) = 0 \tag{2.12}$$

Equation (2.12) establishes the relation between the radius a of the area of contact, the penetration of the die w_0 and the strain-hardening power μ. Equation (2.3) gives one further equation relating a, w_0, P, A and μ.

Evaluation of the integrals occurring in (2.10), (2.11) for some dies of practical interest such as conical, spherical, parabolic dies is difficult in view of the power μ of the function $w(r)$ in (2.11). In order to facilitate the evaluation of the integrals we make the approximation of expressing the function $f(r) = w^{\mu}(r)$ as a polynomial.

Consider the case of a die with a reasonably smooth shape. We shall suppose that the function $\varphi(r)$ has continuous first and second derivatives within the range $[0, l]$, where $l > a$. Then the function $f(r)$ has the same property. Consider the function $\lambda(t) = f''(\sqrt{t})$, which is continuous within the range $[0, l^2]$. By the theorem of Weierstrass we can approximate to this function to any degree of accuracy by the polynomial*

$$q(t) = \sum_{i=0}^{k} a_i t^i$$

We can then approximate to the function $f''(r)$ by means of the polynomial

$$q(r^2) = \sum_{i=0}^{k} a_i r^{2i}$$

which contains only even powers of r. It is then easily seen that the polynomials

$$Q(r) = \sum_{i=0}^{k} \frac{a_i}{(2i+1)(2i+2)} r^{2i+2} + f'(0)\, r + w_0^{\mu}$$
$$Q'(r) = \sum_{i=0}^{k} \frac{a_i}{2i+1} r^{2i+1} + f'(0) \qquad (2.13)$$

are approximations for $f(r)$ and $f'(r)$, respectively, to an accuracy proportional to the accuracy of approximation of the polynomial $q(r^2)$ for the function $f''(r)$. It can easily be verified that the value of the pressure found from these approximations is very close to the true value.

Substituting (2.13) into (2.11), we obtain

$$\psi(u) = w_0^{\mu} + \frac{\sqrt{\pi}}{2}\frac{\Gamma(1-\mu/2)}{\Gamma(1.5-\mu/2)} f'(0)\, u + \sum_{i=0}^{k} \frac{a_i b_i}{2i+1} u^{2i+2}$$

$$b_i = \frac{1}{2}\,\frac{i!}{(1-\mu/2+i)(1-\mu/2+i-1)\dots(1-\mu/2+1)(1-\mu/2)}$$

$$p(r) = -Ac^{-\mu}a^{1-\mu}\frac{\sin \pi\mu/2}{\pi^2}\left[\frac{\sqrt{\pi}}{2}\frac{\Gamma(1-\mu/2)}{\Gamma(1.5-\mu/2)} f'(0)\int_{\rho}^{1}\frac{dt}{(t^2-\rho^2)^{\mu/2}} + \sum_{i=0}^{k}\frac{a_i b_i a^{2i+1}}{(2i+1)(2i+2)}\int_{\rho}^{1}\frac{t^{2i+1}dt}{(t^2-\rho^2)^{\mu/2}}\right]$$

where $\rho = r/a$, $\Gamma(x)$ is the gamma-function. The second integral in the

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* For such a polynomial we could take [6]

$$\sum_{i=0}^{k} C_k^{\,i}\left(\frac{t}{l^2}\right)^i\left(1-\frac{t}{l^2}\right)^{k-i}\lambda\left(\frac{i}{k}\,l^2\right)$$

expression for $p(r)$ can be integrated by parts; we have

$$\int_{\rho}^{1} \frac{t^{2i+1}dt}{(t^2-\rho^2)^{\mu/2}} = (1-\rho^2)^{1-\mu/2}\Big[\frac{1}{1-\mu+2i+2-1} + \frac{2i\rho^2}{(-\mu+2i+2)(-\mu+2i+2-3)} + \ldots + \frac{(2i)!!\,\rho^{2i}}{(-\mu+2i+2)(-\mu+2i+2-3)\ldots(-\mu+3)(-\mu+1)}$$

and the first integral in the expression for $p(r)$ must be integrated by a tabular method. In the case of dies which do not come to a point at $r = 0$, $f'(0) = 0$ and the term containing the first integral in the expression for $p(r)$ disappears. Condition (2.12) enables us to express the radius a of the area of contact in terms of the settlement of the die w_0. Condition (2.3) gives

$$P = Ac^{-\mu}(\mu)\, h(w_0, \mu)$$

Note that the unknown quantity $c(\mu)$ (constant for a given material) appearing in the relation between the force and the settlement of the die, can be found from penetration tests with any single die.

Sometimes penetration tests on solid bodies are used for the experimental determination of the mechanical properties of a material. With the aid of such experiments the solutions derived in this paper enable the strain-hardening power to be determined. For instance, if values are known for the force and settlement for two penetrations of a die with a plane circular base, then Formula (2.9) gives the following equation for μ;

$$\frac{P_1}{P_2} = \left(\frac{w_{01}}{w_{02}}\right)^{\mu}$$

In the case of a cone or sphere, the corresponding equation is (2.12), or the equation

$$\frac{P_1}{P_2} = \frac{h(w_{01}, \mu)}{h(w_{02}, \mu)}$$

Finally, note that the results given above can be applied to the case of steady and quasi-steady creep, which is described by the equations of the yield theory [7] (assuming incompressibility of the material and a power-law relation between the intensity of shear-strain rate and intensity of shear stress).

3. Penetration of a die into a half-space with nonsteady creep of the material. We shall assume that the creep of an incompressible material is described by the equations proposed by

Rabotnov [8]:

$$A\Gamma^{\mu-1}(t)\,\varepsilon_{ij}(t) = \sigma_{ij}'(t) - \int_0^t K(t-\tau)\,\sigma_{ij}'(\tau)\,d\tau \qquad (3.1)$$

Here t is the time (for brevity the three-dimensional variables have been discarded), σ_{ij}' is the stress deviator, Γ is the intensity of shear strain, A and μ are constants of the material $(0 < \mu \leqslant 1)$, $K(t - \tau)$ is the relaxation kernel.*

If the application of the load is instantaneous, the material at the moment $t = 0$ is governed by Equation (3.1) and behaves as if it were subject only to (power-law) strain-hardening.

Consider the quasi-statical problem of a concentrated force $P(t)$ applied normally to the boundary of a half-space. Since the operator in the right-hand side of (3.1) is linear and homogeneous, and since the three-dimensional variables appear in (3.1) as parameters, all the ideas of Section 1 concerning the dependence of the unknowns on the radius R still hold and can be applied to the present problem. In addition to the angle θ, however, the time t must also be included in the arguments of functions ζ, e_{ij}, g, s_{ij}. Solving Equations (3.1) for σ_{ij}' and making use of (1.20), we obtain

$$s_{ij}(\theta, t) = Ag^{\mu-1}(\theta, t)\,e_{ij}(\theta, t) + \int_0^t N(t-\tau)\,Ag^{\mu-1}(\theta, \tau)\,e_{ij}(\theta, \tau)\,d\tau \quad (3.2)$$

where $N(t - \tau)$ is the resolvent of the kernel $K(t - \tau)$.

In order to obtain an equation for the function $\zeta(\theta, t)$ we substitute Formulas (1.15) into (3.2) and then substitute the resulting expression for s_{ij} into the differential equation (1.23). Substituting (3.2) into (1.23) and taking the differential operator on the left-hand side of Equation (1.23) under the integral sign, we obtain a homogeneous Volterra integral equation

$$u(\theta, t) + \int_0^t N(t-\tau)\,u(\theta, \tau)\,d\tau = 0 \qquad (3.3)$$

in the function $u(\theta, t) = U\zeta$, where U is the differential operator of Equation (1.25). It follows from (3.3) that $u(\theta, t) = 0$, i.e. the function $\zeta(\theta, t)$, satisfies the ordinary differential equation (1.25).

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* Without prejudicing the theory that follows, the kernel $K(t - \tau)$ can be replaced by the more general form $K(t, \tau)$, and the lower limit of integration can be replaced by the constant t_0.

We now substitute (3.2) into the boundary conditions (1.27). In this way we obtain two homogeneous integral equations of the same type as (3.3): the first in $[g^{\mu-1}e_{R\theta}]_{\theta=\pi/2}$, the second in $[g^{\mu-1}(e_{R\theta}' + e_\theta - e_\varphi)]_{\theta=\pi/2}$. It follows from these equations that the function $\zeta(\theta, t)$ satisfies the boundary conditions (1.29). The boundary condition (1.30) evidently still applies to the case under discussion. Substituting (3.2) into condition (1.31) we obtain

$$v(t) + \int_0^t N(t-\tau)\, v(\tau)\, d\tau = \frac{P(t)}{\pi A}$$

Solving this integral equation for $v(t) = V[\zeta]$ we find that

$$V[\zeta] = \frac{1}{\pi A}(1-L)\,P(t), \qquad \left(Ly(t) = \int_0^t K(t-\tau)\, y(\tau)\, d\tau\right) \tag{3.4}$$

Thus the function $\zeta(\theta, t)$ satisfies the same differential equation (1.25) and the same boundary conditions (1.29), (1.30) as in the corresponding problem with strain-hardening which follows a power-law. Condition (3.4) differs from (1.32) only in the different value of the right-hand side. Note that the time t appears in the equation and in the condition for the function $\zeta(\theta, t)$ as a parameter. Consequently, the solution of the problem of a concentrated force can be found from the solution of the problem of Section 1, by replacing the force P by the quantity $(1 - L)P(t)$.

We proceed now to the problem of the penetration of a rigid die into a half-space, the properties of the material of which are described by Equations (3.1). We shall assume that there is no friction on the area of contact. As in Section 2, applying the principle of superposition of the "generalized displacement" w^μ, we obtain the following expression for determining the pressure under the die:

$$\iint_{(S_t)} \frac{(1-L)\,p(x_1, y_1, t)\, dx_1 dy_1}{\left(\sqrt{(x-x_1)^2+(y-y_1)^2}\right)^{2-\mu}} = Ac^{-2}w^\mu(x, y, t) \tag{3.5}$$

Here (S_t) is the area of contact; the function $w(x, y, t)$ is given by (2.1), in which the quantities α, β, w_0, determined from the conditions (2.3), (2.4), are, in general, functions of time. Equation (3.5) is equivalent to the following two equations analogous to the corresponding equations in the plane problem [2]:

$$\omega(x, y, t) - \int_0^t K(t-\tau)\, \omega(x, y, \tau)\, d\tau = w^\mu(x, y, t) \tag{3.6}$$

$$\iint_{(S_t)} \frac{p(x_1, y_1, t)\, dx_1 dy_1}{\left(\sqrt{(x-x_1)^2 + (y-y_1)^2}\right)^{2-\mu}} = Ac^{-\mu}\omega(x, y, t) \tag{3.7}$$

Equation (3.6) is a Volterra integral equation of the second kind (x, y occur in this equation as parameters).

Equation (3.7) is a Fredholm equation of the first kind (the variable t occurs in (3.7) as a parameter). The constant $c(\mu)$ can be determined by short-duration penetration tests with a die.

Consider the penetration into a half-space of a die with a flat base under the action of a force $P(t)$. In this case the area of contact is fixed and $w = w_0{}^{\mu}(t)$ is independent of x, y. It follows from (3.6) that $\omega = \omega_0(t)$

$$\omega_0(t) = w_0{}^{\mu}(t) + \int_0^t N(t-\tau)\, w_0{}^{\mu}(\tau)\, d\tau \tag{3.8}$$

is also independent of x, y. But then

$$p(x, y, t) = p_1(x, y)\, Ac^{-\mu}\omega_0(t) \tag{3.9}$$

Here $p_1(x, y)$ is the solution of Equation (3.7) when the absolute term is unity. After eliminating $Ac^{-\mu}\omega_0(t)$ by expressing it in terms of $P(t)$ with the aid of (2.3), we obtain the pressure distribution

$$p(x, y, t) = P(t)\, p_1(x, y) \left(\iint_{(S)} p_1(x, y)\, dx dy\right)^{-1}$$

which coincides with the pressure distribution for the case of power-law strain-hardening (or for the case of instantaneous penetration). Thus for a die with a flat base under the action of a force, creep has no effect on the law governing the pressure distribution under the die. This result is completely analogous to the corresponding result for the plane problem [2]. Thus, for a die with a plane elliptical base the pressure distribution is given by the first of Formulas (2.7), in which P is considered as a known function of time; then in order to find the relation between the force P and the settlement w_0 the quantity $w_0{}^{\mu}$ in the second of Formulas (2.7) should be replaced by ω_0 as given by (3.8).

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Translated by J.K.L.